

# Large tick assets: implicit spread and optimal tick size

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## Abstract

In this work, we bring to light a quantity, referred to as *implicit spread*, playing the role of spread for large tick assets, for which the effective spread is almost always equal to one tick. The relevance of this new parameter is shown both theoretically and numerically. This implicit spread allows us to quantify the tick sizes of large tick assets and to define a notion of *optimal tick size*. Moreover, our results open the possibility of forecasting the behavior of relevant market quantities after a change in the tick value and to give a way to modify it in order to reach an optimal tick size.

**Key words:** Microstructure of financial markets, high frequency data, large tick assets, implicit spread, market making, limit orders, market orders, optimal tick size.

# 1 Introduction

## 1.1 Tick value, tick size and spread

On electronic markets, the market platform fixes a grid on which traders can place their prices. The grid step represents the smallest interval between two prices and is called the *tick value* (measured in the currency of the asset). For a given security, it is safe to consider this grid to be evenly spaced even though the market may change it at times. In some markets the spacing of the grid can depend on the price. For example, stocks trading on the Euronext in Paris have a price dependent tick scheme. Stocks priced 0 to 9.999 euros have a tick value of 0.001 euro but all stocks above 10 euros have a tick value of 0.005 euro.

However, when it comes to actual trading, the tick value is given little consideration. What is important is the so called *tick size*. A trader considers that an asset has a small tick size when he “feels” it to be negligible, in other words, when he is not averse to price variations of the order of a single tick. In general then, the trader’s perception of the tick size is qualitative and empirical, and depends on many parameters such as the tick value, the price, the usual amounts traded in the asset and even his own trading strategy. All this leads to the following well known remark : the tick value is not a good absolute measure of the perceived size of the tick. It has to be viewed relatively to other market statistics. For instance, every trader “considers” that the ESX index futures has a much greater tick than the DAX index futures though their tick values have the same order of magnitude.

Nevertheless, the notion of “large tick asset” is rather well understood. For Eisler, Bouchaud and Kockelkoren [14]: “large tick stocks are such that the bid-ask spread is almost always equal to one tick, while small tick stocks have spreads that are typically a few ticks”. We borrow this definition in this work. This type of assets lead to the following specific

issues which we address in this paper:

- How to quantify more precisely the tick size of large tick assets ?
- Many studies have pointed out special relationships between the spread and some market quantities. However, these studies reach a limit when discussing large tick assets since the spread is artificially bounded from below. How to extend these studies to this kind of assets ?
- What happens to the relevant market quantities when the market designer decides to change the tick value and what is then the optimal tick value ?

One of the aim of this paper is therefore to understand if some notion of spread can be defined for large tick assets. Before that, we recall two approaches leading to important relationships between the spread and other market quantities in the case of small tick assets.

## 1.2 The Madhavan *et al.* spread theory for small tick assets

The way the spread is fixed on the market is widely studied in the microstructure literature, see for example [8, 16, 17, 24, 25, 26, 27, 30, 32, 41]. In particular, several theoretical models have been built in order to understand the determinants of the spread, see [9, 13, 15, 29, 39, 40]. Here we give a here a brief, simplified, overview of Madhavan, Richardson, Rooman seminal paper [31] on the link between spread and volatility. In [31], the authors assume the existence of a *true* or *efficient price* for the asset with ex post value  $p_i$  after the  $i^{\text{th}}$  trade and that all transactions have the same volume. Then they consider the following dynamic for the efficient price:

$$p_{i+1} - p_i = \xi_i + \theta \varepsilon_i,$$

with  $\xi_i$  an iid centered shock component (new information, . . .) with variance  $v^2$ ,  $\varepsilon_i$  the sign of the  $i^{\text{th}}$  trade and  $\theta$  an impact parameter. Note that, in order to simplify the presentation, we assume here that the  $\varepsilon_i$  are independent (in [31], the authors allow for short term dependence in the  $\varepsilon_i$ ).

The idea in [31] is then to consider that market makers cannot guess the surprise of the next trade. So, they post (pre trade) bid and ask prices  $a_i$  and  $b_i$  given by

$$a_i = p_i + \theta + \varphi, \quad b_i = p_i - \theta - \varphi,$$

with  $\varphi$  an extra compensation claimed by market makers, covering processing costs and the shock component risk. The above rule ensures no ex post regret for the market makers: If  $\varphi = 0$ , the traded price is on average the right one. In particular, the ex post average cost of a market order with respect to the efficient price  $a_i - p_{i+1}$  or  $p_{i+1} - b_i$  is equal to 0.

The Madhavan *et al.* model allows to compute several relevant quantities. In this approach, we obtain that

- The spread  $S$  is given by  $S = a - b = 2(\theta + \varphi)$ .
- Neglecting the contribution of the news component, see for example [43] for details, the variance per trade of the efficient price  $\sigma_{tr}^2$  satisfies

$$\sigma_{tr}^2 = \mathbb{E}[(p_{i+1} - p_i)^2] = \theta^2 + v^2 \sim \theta^2.$$

- Therefore:

$$S \sim 2\sigma_{tr} + 2\varphi.$$

This last relation, which gives a very precise link between the spread and the volatility

per trade, will be one of the cornerstones of our study.

### 1.3 The Wyart *et al.* approach

We recall now the Wyart *et al.* approach, see [43], which is another way to derive the proportionality between the spread and the volatility per trade. Here again, the idea is to consider a dichotomy between market makers and market takers. Market makers are patient traders who prefer to send limit orders and wait to be executed, thus avoiding to cross the spread but taking on volatility risk. Market takers are impatient traders who prefer to send market orders and get immediate execution, thus avoiding volatility risk but crossing the spread in the process. Wyart *et al.* consider a generic market making strategy and show that its average profit and loss per trade per unit of volume can be well approximated by the formula

$$\frac{S}{2} - \frac{c}{2}\sigma_{tr},$$

where  $S$  denotes the average spread and  $c$  is a constant depending on the assets, but of order  $1 \sim 2$ .

This profit and loss should correspond to the average cost of a market order. Then Wyart *et al.* argue that on electronic market, any agent can chose between market orders and limit orders. Consequently the market should stabilize so that both types of orders have the same average (ex post) cost, that is zero. Indeed, because of the competition between liquidity providers, the spread is the smallest admissible value such that the profit of the market makers is non negative (otherwise another market maker would come with a tighter spread). Thus, if the tick size allows for it, the spread is so that market makers do not make profit.

Therefore, in this case:

$$S \sim c\sigma_{tr}.$$

Moreover, in [43], Wyart *et al.* show that this relationship is very well satisfied on market data.

## 1.4 Aim of this work and organization of the paper

The goal of this work is to bring to light a quantity, referred to as *implicit spread*, playing the role of spread for large tick assets, for which the effective spread is almost always equal to one tick. In particular, it will enable us to quantify the tick sizes of this type of assets and to define a notion of *optimal tick size*. The implicit spread is defined thanks to a statistical model described in Section 2. In order to validate the fact that our new quantity can be seen as a spread for large tick assets, we show in Section 3 the striking validity of the relationship between spread and volatility per trade mentioned above on various electronically traded large tick assets, provided the spread is replaced by the implicit spread. We also explain this relationship through a very simple equilibrium model in Section 4. Finally, in Section 5, we show that these results enable us to forecast the behavior of relevant market quantities after a change in the tick value and to give a way to modify it in order to reach an optimal tick size.

## 2 The model with uncertainty zones

The implicit spread can be naturally explained in the framework of the model with uncertainty zones developed in [34]. Note that we could introduce this notion without referring to this model. However, using it is very convenient in order to give simple intuitions.

## 2.1 Statistical model

The model with uncertainty zones is a model for transaction prices and durations (more precisely, only transactions leading to a price change are considered). It is a *statistical model*, which means it has been designed in order to reproduce the stylized facts observed on the market and to be useful for practitioners. It is shown in [34] that this model indeed reproduces (almost) all the main stylized facts of prices and durations at any frequency (from low frequency data to ultra high frequency data). In practice, this model is particularly convenient in order to estimate relevant parameters such as the volatility or the covariation at the ultra high frequency level, see [35], or when one wants to hedge a derivative in an intraday manner, see [33].

A priori, such a model is not firmly rooted on individual behaviors of the agents. However since it reproduces what is seen on the market, the way market participants acts has to be consistent with the model. Therefore, as explained in the rest of this section and in Section 4, ex post, an agent based interpretation of such a statistical model can still be given.

## 2.2 Description of the model

The heuristic of the model is very simple. When the bid-ask is given, market takers know the price for which they can buy and the price for which they can sell. However, they have their own opinion about the efficient price of the asset, inferred from all available market data and their personal views. In the latter, we assume that there exists an efficient price, representing this opinion. Of course this efficient price should not be seen as an “economic price” of the asset, but rather as a market consensus at a given time about the asset value. The idea of the model with uncertainty zones is that for large tick assets, at a given time  $t$ , the difference between the efficient price and the best accessible price on the market for

buying (resp. selling) is sometimes too large so that a buy (resp. sell) market order can occur.

### 2.2.1 Efficient price

We propose here a simplified version of the model with uncertainty zones, see [34] for a more general version. The first assumption on the model is the following:

H1 *There is a latent efficient price with value  $X_t$  at time  $t$  of the form  $X_t = X_0 + \sigma W_t$ , where  $W_t$  is a Brownian motion and  $\sigma > 0$  is the volatility parameter.*

Following in particular the works by Aït-Sahalia *et al.*, see [3, 45], using a Brownian type efficient price when building a microstructure model has become very popular in the recent financial econometrics literature. Indeed, it enables to easily retrieve standard Brownian type dynamics in the low frequencies, which is in agreement with both the behavior of the data and the classical mathematical finance theory.

Even though the model allows for more complex dynamics for the efficient price, to simplify, we assume that it is simply a Brownian motion (in particular since we are working at the high frequency level, we can neglect the drift and we do not need to consider an exponential form for the price). Of course this efficient price is not observed by market participants. However, they may have their own opinion about its value.

### 2.2.2 Uncertainty zones and dynamics of the last traded price

Let  $\alpha$  be the tick value of the asset. We define the uncertainty zones as bands around the mid tick values with width  $2\eta\alpha$ , with  $0 < \eta \leq 1$  a given parameter. The dynamics of the last traded price, denoted by  $P_t$ , is obtained as a functional of the efficient price and the uncertainty zones. Indeed, in order to change the transaction price, we consider that market

takers have to be “convinced” that it is reasonable, meaning that the efficient price must be close enough to a new potential transaction price. This is translated in Assumption H2.

H2 *When  $P_t = b$ , one cannot have a transaction at price  $b + \alpha$  until  $X_t$  upcrosses the uncertainty zone above  $b$ , that is hits the value  $b + \alpha/2 + \eta\alpha$ . Symmetrically, if  $P_t = b + \alpha$ , one cannot have a transaction at price  $b$  until  $X_t$  downcrosses the uncertainty zone below  $b + \alpha$ , that is hits the value  $b + \alpha/2 - \eta\alpha$ . Moreover, when  $P_t = b$  (resp.  $P_t = b + \alpha$ ) one has a transaction at price  $b + \alpha$  (resp.  $b$ ) at the first time  $X_t$  crosses the value  $b + \alpha/2 + \eta\alpha$  (resp.  $b + \alpha/2 - \eta\alpha$ ).*

In fact, when associating it to Assumption H3, we will see that Assumption H2 can be understood as follows: for any given time  $t$ , a buy market order cannot occur if the efficient price is too far from the best ask and conversely for a sell market order.

Remark that Assumption H2 implies that the transaction price only jumps by one tick, which is fairly reasonable for large tick assets. However, imposing jumps of only one tick and that a transaction occurs exactly at the time the efficient price exits the uncertainty zones is done only for technical convenience. Indeed, it can be easily relaxed in the setting of the model with uncertainty zones, see [34].

A sample path of the last traded price in the model with uncertainty zones is given in Figure 1.

### 2.2.3 Bid-ask spread

In this work, we focus on “large tick assets”. By this we mean assets whose bid-ask spread is essentially constant and equal to one tick. Therefore we make the following assumption in the model.

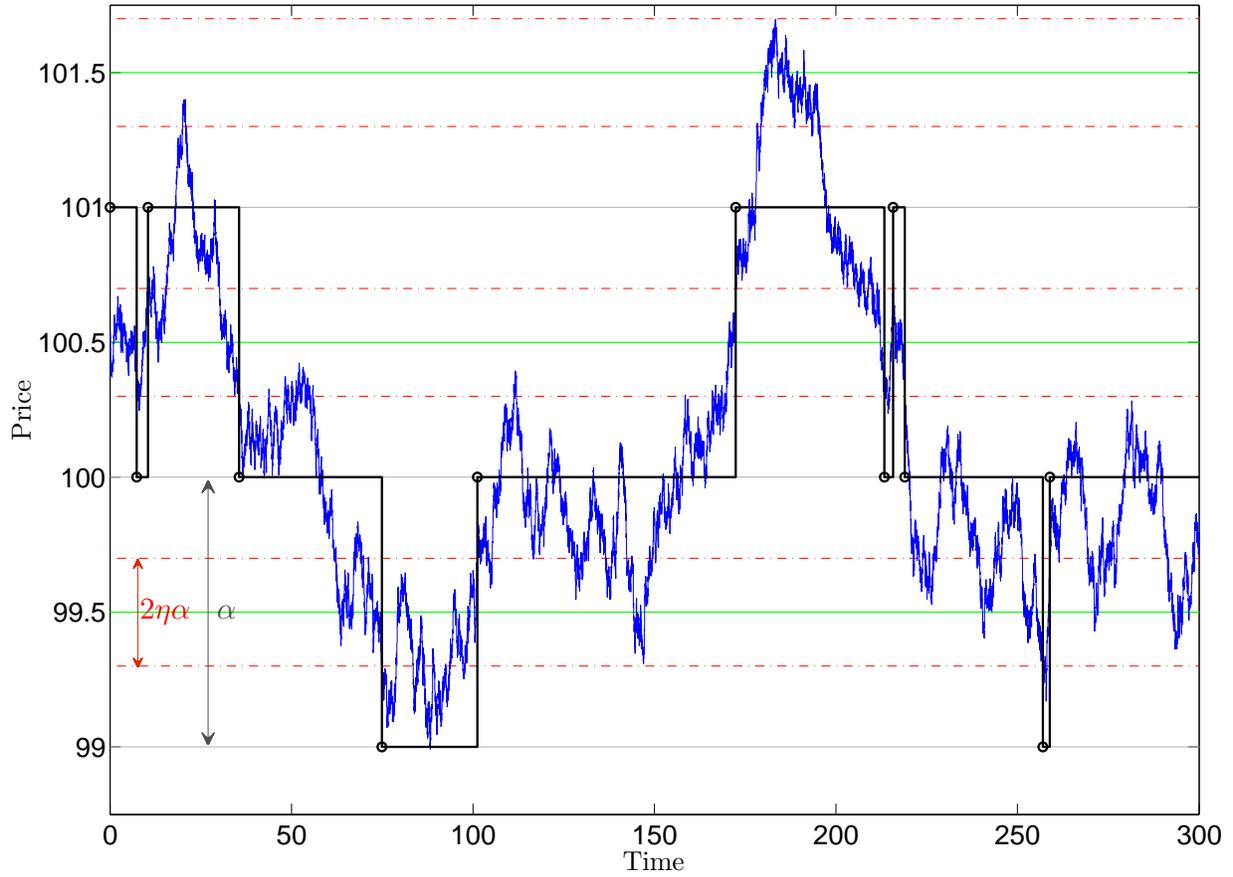


Figure 1: The model with uncertainty zones: example of sample path. The efficient price is drawn in blue. The light gray lines drawn at integers form the tick grid of width  $\alpha$ . The red dotted lines are the limits of the uncertainty zones of width  $2\eta\alpha$ . Finally the last traded price is the black stepwise curve. The circles indicate a change in the price when the efficient price crosses the uncertainty zone.

H3 *The bid-ask spread is constant, equal to the tick value  $\alpha$ .*

In practice, the preceding assumption means that if at some given time  $t$  the spread is not equal to one tick, limit orders immediately fill the gap. Remark that we do not impose the efficient price to lie inside the bid-ask quotes. However, the dynamics of the bid-ask quotes still need to be compatible with Assumption H2.

Within bid-ask quotes of the form  $[b, b + \alpha]$ , the width of the uncertainty zone represents the range of values for  $X_t$  where we can have both transactions at the best bid and the best ask. The size of this range is  $2\eta\alpha$ . Therefore, it is natural to view the quantity  $2\eta\alpha$  as an *implicit spread*, see Section 3. More precisely, for given bid-ask quotes  $[b, b + \alpha]$ , Assumptions H2 and H3 enable us to define three areas for  $X_t$ :

- The bid zone:  $(b - \alpha/2 - \eta\alpha, b + \alpha/2 - \eta\alpha)$ , where only sell market orders occur.
- The buy/sell zone:  $[b + \alpha/2 - \eta\alpha, b + \alpha/2 + \eta\alpha]$ , where both buy and sell market orders can occur. It coincides with the uncertainty zone.
- The ask zone:  $(b + \alpha/2 + \eta\alpha, b + 3\alpha/2 + \eta\alpha)$ , where only buy market orders can occur.

This is summarized in Figure 2.

### 2.3 Comments on the model and the parameter $\eta$

- The model is particularly parsimonious since it only relies on two parameters: the volatility  $\sigma$  and the uncertainty zones parameter  $\eta$ . Again, we stress the fact that despite its simplicity, this simple model accurately reproduces the stylized facts of market data, see [34].

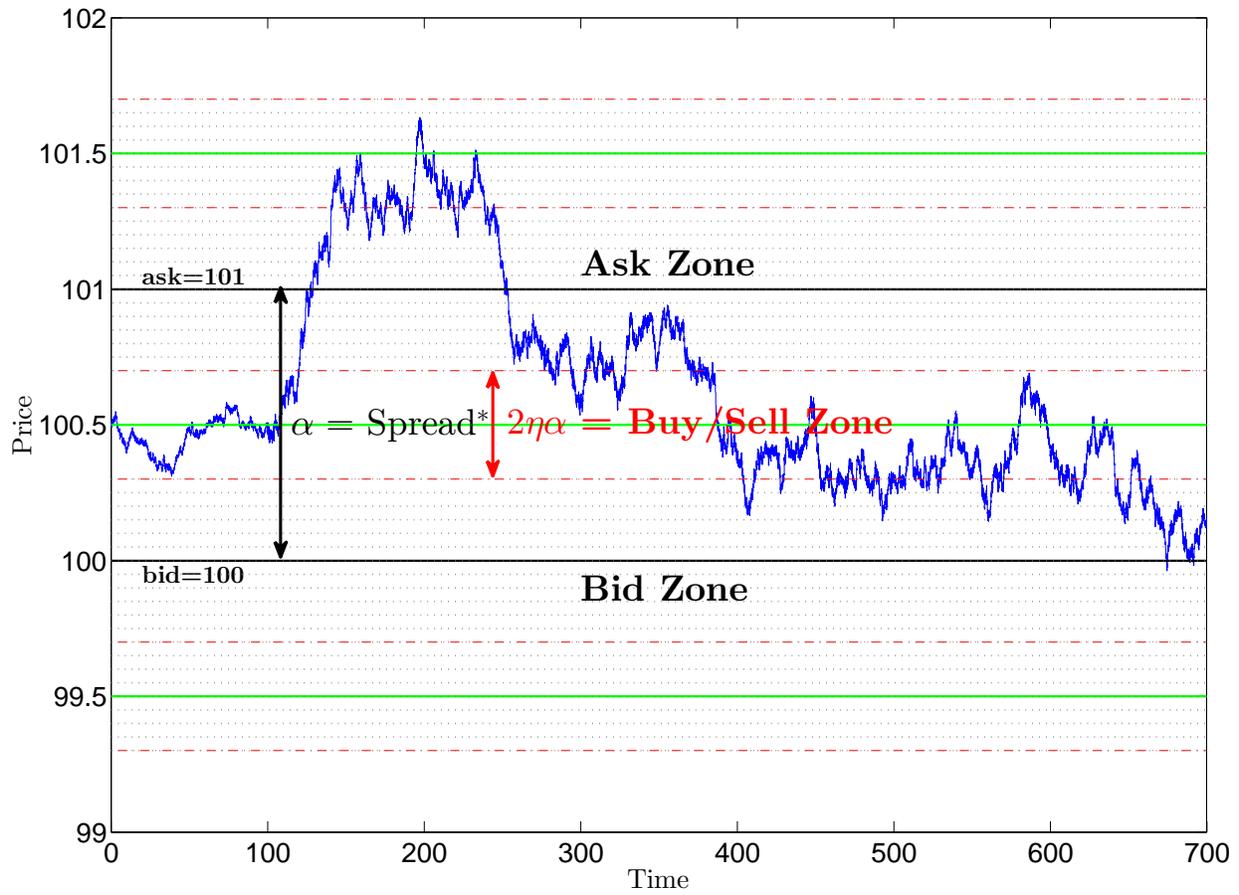


Figure 2: The three different zones when the bid-ask is 100-101 and the tick value is equal to one. The red dotted lines are the limits of the uncertainty zones. The uncertainty zone inside the spread is the buy/sell zone. The upper dotted area is the ask zone and the lower dotted area is the bid zone.

- The parameter  $\eta$  can be very easily estimated as follows. We define an alternation (resp. continuation) of one tick as a price jump of one tick whose direction is opposite to (resp. the same as) the one of the preceding price jump. Let  $N_{\alpha,t}^{(a)}$  and  $N_{\alpha,t}^{(c)}$  be respectively the number of alternations and continuations of one tick over the period  $[0, t]$ . It is proved in [35] that as the tick value goes to zero, a consistent estimator of  $\eta$  over  $[0, t]$  is given by

$$\widehat{\eta}_t = \frac{N_{\alpha,t}^{(c)}}{2N_{\alpha,t}^{(a)}}.$$

- When the tick size is large, market participants are not indifferent to a one tick price change and the traded price is modified only if market takers are convinced it is reasonable to change it. This is exactly translated in our model through the key parameter  $\eta$ . Indeed, in order to have a new transaction price,  $X_t$  needs to reach a barrier which is at a distance  $\eta\alpha$  from the mid tick. So, when  $\eta$  is small, a very small percentage of the tick value is considered enough for a price change, meaning the tick value is very large and conversely. A different point of view is to consider that market participants have a certain resolution, or precision at which they infer the efficient price  $X_t$ . This resolution is quantified by  $\eta$ , and is close to the tick value when  $\eta$  is close to  $1/2$ .
- The intensity of the microstructure effects is quantified by  $\eta$ . For example, it is shown in [35] that as  $\alpha$  goes to zero,

$$\sum_{0 \leq \tau_i < \tau_{i+1} \leq t} (P_{\tau_{i+1}} - P_{\tau_i})^2 \rightarrow \frac{\sigma^2 t}{2\eta}, \quad (1)$$

where the  $\tau_i$  denote the transaction times. Therefore, if  $\eta < 1/2$ , we recover here the very well known stylized fact that the high frequency realized variance of the observed

price is larger than those of the efficient price which is  $\sigma^2$ . More precisely, in that case, we obtain a decreasing behavior of the so called signature plot, that is the function from  $\mathbb{N}^*$  to  $\mathbb{R}^+$  defined by

$$\Delta \rightarrow \sum_{i=1}^{\lfloor nt/\Delta \rfloor} (P_{\Delta i/n} - P_{\Delta(i-1)/n})^2, \quad (2)$$

where  $n$  is a fixed ultra high frequency sampling value for the last traded price. Since the seminal paper [4], this is considered in the econometric literature as one of the most distinctive features of high frequency data. In fact, the estimated values of  $\eta$  are indeed systematically found to be smaller than  $1/2$ . This can be explained from a theoretical point of view, see Section 4. Finally, note that the convergence (1) enables to easily estimate the cumulative variance over  $[0, t]$ ,  $\sigma^2 t$ , by

$$\widehat{\sigma}_t^2 = 2\eta \sum_{0 \leq \tau_i < \tau_{i+1} \leq t} (P_{\tau_{i+1}} - P_{\tau_i})^2. \quad (3)$$

- The width of the buy/sell zone is  $2\eta\alpha$ . Thus, if  $\eta$  is small, there is a lot of mean reversion in the price and the buy/sell area is very small: the tick size is very large. If  $\eta$  is close to  $1/2$ , the last traded price can be seen as a sampled Brownian motion, there is no microstructure effects and the buy/sell area is equal to one tick: the tick size is, in some sense, *optimal*, see Section 5.
- In fact, we can give a much more precise interpretation of  $\eta$ . Indeed, we show in the next section that the quantity  $2\eta\alpha$  can be seen as an implicit spread. A by product of this is the fact that  $\eta$  can indeed be viewed as a suitable measure for the tick size.

### 3 Implicit spread and volatility per trade: empirical study

A buy/sell area  $[b + \alpha/2 - \eta\alpha, b + \alpha/2 + \eta\alpha]$  is a kind of a frontier, such that crossing it makes market takers change their view on the efficient price. It is a sort of tolerance zone defined by their risk aversion to losing one tick. The width of this zone,  $2\eta\alpha$ , also corresponds to the size of the (efficient) price interval for which market takers are both ready to buy and to sell. This is why we see it as a kind of a spread: *the market taker's implicit spread*. In view of this interpretation, we consider the similarities in the properties of this implicit spread to those of the conventional spread. In particular, we look at the spread-volatility relationship described in Section 1 that stipulates that the spread is generally proportional to the volatility per trade. In this section, we empirically verify this relationship using our implicit spread and see that it holds remarkably well.

This approach follows in the some sense those of Roll in [36]. In this paper, the author addresses the problem of estimating the bid-ask spread if one has only access to transaction data. He shows that in his framework, the quantity  $\sqrt{-2\text{Cov}}$ , where Cov denotes the first order autocovariance of the price increments, is a good proxy for the spread. This is particularly interesting since this autocovariance can be expressed in term of  $\eta$ . Indeed, in the model with uncertainty zones, we have

$$\sqrt{-2\text{Cov}} = \sqrt{\frac{2 - 4\eta}{1 + 2\eta}}\alpha.$$

Thus the link between a parameter such as  $\eta$  and a kind of spread is already present in [36]. However, in [36], the author works at a completely different time scale and this measure is not

relevant for large tick assets traded at high frequency on electronic markets. In particular, it decreases with  $\eta$  for  $\eta$  between 0 and 1/2, which is not consistent with the numerical experiments.

### 3.1 Definition of the variables

In this section, we want to investigate the relationship

$$\text{Implicit spread} \sim \text{Volatility per trade} + \text{constant}.$$

The implicit spread and the volatility per trade are computed on a daily basis. Following the approach of Madhavan *et al.* [31], the volatility over the period is taken with reference to the efficient price. We use the estimator  $\widehat{\sigma}_t^2$  of the cumulative variance of the efficient price over  $[0, t]$  introduced in Equation (3) and set

$$\widehat{\sigma} = \sqrt{\widehat{\sigma}_t^2}.$$

Then we define the volatility per trade as  $\widehat{\sigma}/\sqrt{M}$ , where  $M$  denote the number of trades over  $[0, t]$ . From now on, abusing notation slightly, we make no difference between the parameters and their estimators. Therefore, our relationship can be rewritten

$$\eta\alpha \sim \frac{\sigma}{\sqrt{M}} + \varphi.$$

In the sequel we also need to compute an average daily spread, denoted by  $S$ , which is in practice not exactly equal to one. This spread is measured as the average over the considered time period of the observed spreads right before the trades. Thus, for each asset, we record

everyday the vector  $(\eta\alpha, \sigma, M, S)$ .

## 3.2 Description of the data

We restrict our analysis to assets that trade in well regulated electronic markets that match the framework of the electronic double auction. We use data of 10 futures contracts on assets of different classes that trade in different exchanges. The database<sup>1</sup> has millisecond accuracy and was recorded from 2009, May 15 to 2009, December 31.

On the CBOT exchange, we use the 5-Year U.S. Treasury Note Futures (BUS5) and the futures on the Dow Jones index (DJ). On the CME, we use the forex EUR/USD futures (EURO) and the futures on the SP500 index (SP). On the EUREX exchange, we use three interest rates futures based on German government debt: The 10-years Euro-Bund (Bund), the 5-years Euro-Bobl (Bobl) and the 2-years Euro-Schatz (Schatz). Note that the tick size of the Bobl changed on 2009, June 16. Thus, we write Bobl 1 when referring to the Bobl before this date and Bobl 2 after it. We also investigate futures on the DAX index (DAX) and on the EURO-STOXX 50 index (ESX). Finally we use the Light Sweet Crude Oil Futures (CL) that trades on the NYMEX. As for their asset classes, the DJ, SP, DAX and ESX are equity futures, the BUS5, Bund, Bobl and Schatz are fixed income futures, the EURO is a foreign exchange futures and finally the CL is an energy futures. On the exchanges, the settlement dates for these future contracts are standardized, one every three months (March, June, September and December) and generally three future settlement months are trading at the same time. We deal with this issue by keeping, on each day, the contract that recorded the highest number of trades and discarding the other maturities.

These assets are all large tick assets, with a spread almost always equal to one tick. To

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<sup>1</sup>Data provided by QuantHouse. <http://www.quanthouse.com/>

quantify this, for each asset, we compute everyday the percentage of trades for which the value of the observed spread right before the trade is equal to one tick. The average of these values is denoted by  $\#S_{=}$  and is reported in Table 1, together with other information about the assets, notably the average values of  $\eta$ , denoted by  $\#\eta$ .

Futures	Exchange	Class	Tick Value	Session	# Trades/Day	$\#\eta$	$\#S_{=}$
BUS5	CBOT	Interest Rate	7.8125 \$	7:20-14:00	26914	0.233	94.9
DJ	CBOT	Equity	5.00 \$	8:30-15:15	48922	0.246	81.8
EURO	CME	FX	12.50 \$	7:20-14:00	46520	0.242	90.6
SP	CME	Equity	12.50 \$	8:30-15:15	118530	0.035	99.6
Bobl 1	EUREX	Interest Rate	5.00 €	8:00-17:15	18531	0.268	95.3
Bobl 2	EUREX	Interest Rate	10.00 €	8:00-17:15	11637	0.142	99.2
Bund	EUREX	Interest Rate	10.00 €	8:00-17:15	25182	0.138	98.1
DAX	EUREX	Equity	12.50 €	8:00-17:30	39573	0.275	72.7
ESX	EUREX	Equity	10.00 €	8:00-17:30	35121	0.087	99.5
Schatz	EUREX	Interest Rate	5.00 €	8:00-17:15	9642	0.122	99.4
CL	NYMEX	Energy	10.00 \$	8:00-13:30	73080	0.228	75.7

Table 1: Data Statistics. The *Session* column indicates the considered trading hours (local time). The sessions are chosen so that we get enough liquidity and are not the actual sessions.

### 3.3 Graphical analysis

In order to have a first idea of the relevance of our implicit spread, we give the cloud  $(\eta\alpha\sqrt{M}, \sigma)$  in Figure 3. Each point represents one asset, one day.

The results are quite striking. At the visual level, we obtain a linear relationship between  $\sigma$  and  $\eta\alpha\sqrt{M}$  with the same slope but different intercepts. In particular, Figure 3 looks very similar to the kind of graphs obtained in [43], where the real spread is used with small tick assets.

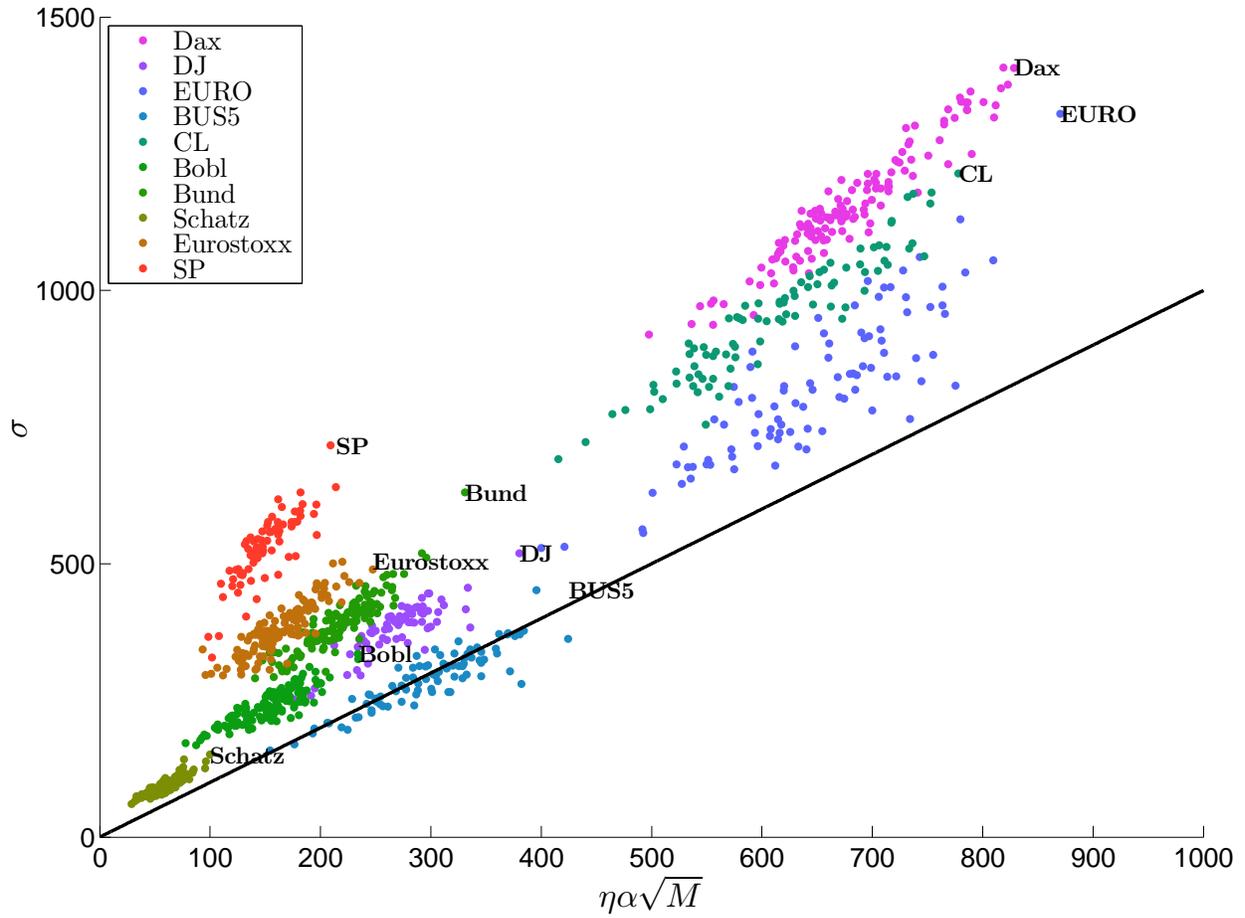


Figure 3: Cloud  $(\eta\alpha\sqrt{M}, \sigma)$ . The black line is the line  $y = x$ .

### 3.4 Linear regression

In order to get a deeper analysis of the relationship between the implicit spread and the volatility per trade, we consider the linear regression associated to the relation

$$\eta\alpha \sim \frac{\sigma}{\sqrt{M}} + \varphi.$$

The constant  $\varphi$  includes costs and profits related to the inventory control and to the fact that the average spread of the assets is not exactly equal to one tick. In the spirit of the approaches mentioned in Section 1, this last fact should imply that the profit of the market makers is slightly larger than in the case where the spread is exactly equal to one tick. Therefore, we consider that for each asset the constant  $\varphi$  is proportional to the average daily spread. Thus we consider the daily regression with unknown  $p_1, p_2, p_3$ :

$$\sigma = p_1\eta\alpha\sqrt{M} + p_2S\sqrt{M} + p_3. \quad (4)$$

The results are given in Table 2.

Asset	$p_1$	$p_2$	$p_3$	$R^2$
BUS5	0.67 [0.55,0.79]	0.10 [0.06,0.14]	-40.21 [-76.28,-4.14]	0.84
DJ	0.93 [0.71,1.15]	0.07 [0.01,0.13]	38.90 [-18.19,96.00]	0.73
EURO	1.31 [1.11,1.51]	0.02 [-0.02,0.07]	-89.23 [-211.08,32.62]	0.75
SP	1.67 [1.37,1.96]	0.07 [0.05,0.08]	-2.84 [-69.90, 64.21]	0.83
Bobl	0.91 [0.84,0.97]	0.08 [0.07,0.09]	19.04 [4.41,33.67]	0.90
Bund	1.11 [1.01,1.20]	0.11 [0.09,0.13]	-29.99 [-54.16,-5.82]	0.92
Dax	1.09 [1.01,1.16]	0.11 [0.10,0.13]	54.94 [23.02,86.86]	0.97
ESX	0.89 [0.78,1.01]	0.13 [0.11,0.15]	-10.15 [-37.71,17.41]	0.90
Schatz	0.80 [0.71,0.90]	0.10 [0.07,0.12]	-0.93 [-9.78,7.92]	0.88
CL	0.97 [0.89,1.05]	0.11 [0.09,0.12]	-11.14 [-51.20,28.92]	0.97

Table 2: Estimation of the linear model with 95% confidence intervals.

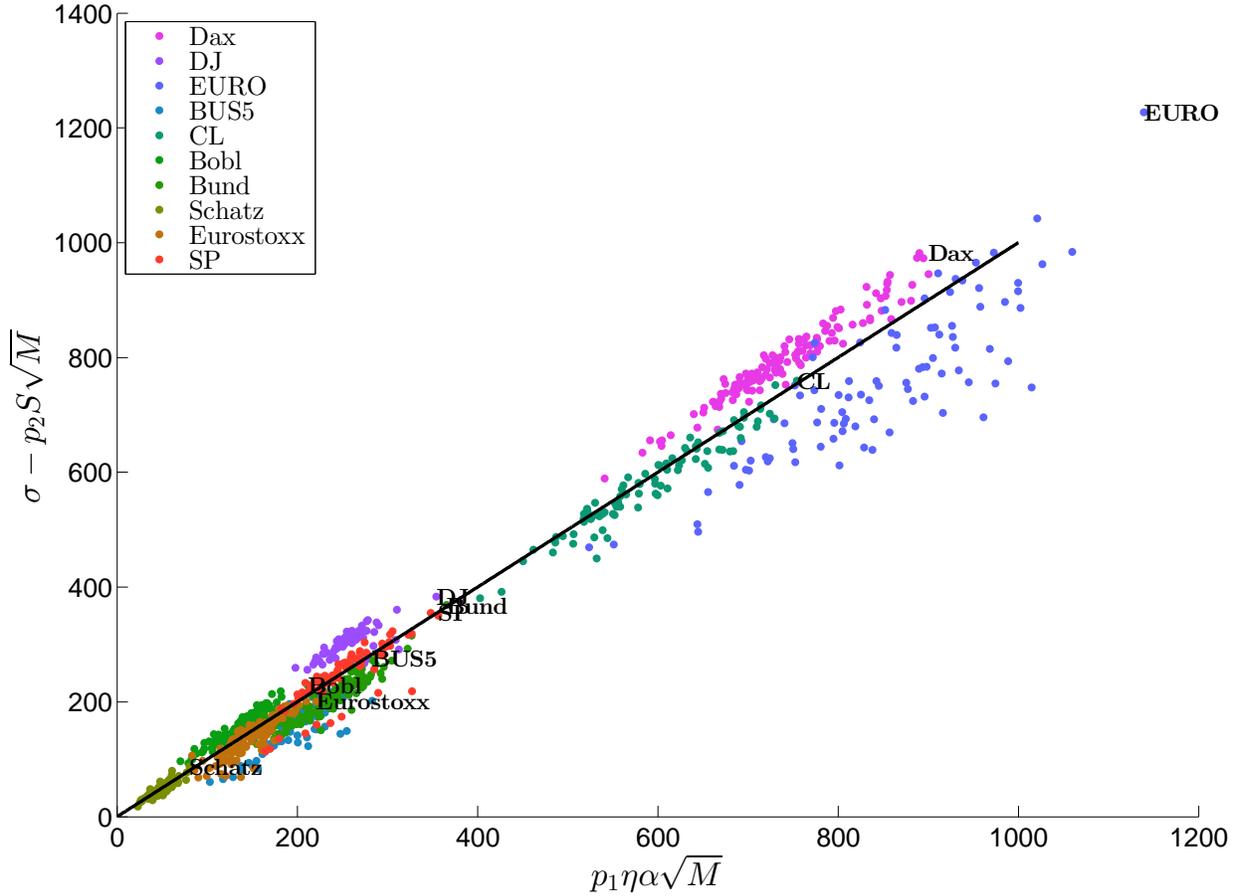


Figure 4: Cloud  $(p_1\eta\alpha\sqrt{M}, \sigma - p_2S\sqrt{M})$ . The black line is the line  $y = x$ .

By looking at the  $R^2$  statistics, we can notice that the linear fits are very good. More interestingly, we see that the values of  $p_1$  are systematically very close to 1. We explain this from a theoretical point of view in the next section. Surprisingly enough, we also remark that the constant  $p_2$  has the same order of magnitude for all the assets (about 0.1). Finally, in order to show that the parameter  $p_3$  is negligible, the cloud  $(p_1\eta\alpha\sqrt{M}, \sigma - p_2S\sqrt{M})$  is given in Figure 4. On this figure, all the points are indeed very close to the line  $y = x$ .

## 4 Implicit spread and volatility per trade: a simple equilibrium model

In our approach, the relationship between the implicit spread  $\eta\alpha$  and the volatility per trade can be theoretically justified in a very natural way. Indeed, we use a simple equilibrium equation between profits and losses of market makers and market takers. To do so, in the spirit of Madhavan *el al.* [31], we compute the ex post expected cost relative to the efficient price of a market order.

### 4.1 A profit and loss equality

To fix ideas, let us consider a market order leading to an upward price change at time  $t$ . We write  $X_t$  for the efficient price at the transaction time  $t$  and assume it lies inside the bid-ask quotes at the transaction time. Therefore, from the model, the market order has been done at price  $P_t = X_t + \alpha/2 - \eta\alpha$ . After this transaction, there are two possibilities:

- The next move of the transaction price is a downward move at price  $P_t - \alpha$ : it means the efficient price has crossed the barrier  $X_t - 2\eta\alpha$ . This occurs with probability  $\frac{1}{1+2\eta}$ .
- The next move of the transaction price is an upward move at price  $P_t + \alpha$ : it means the efficient price has crossed the barrier  $X_t + \alpha$ . This occurs with probability  $\frac{2\eta}{1+2\eta}$ .

Therefore, the ex post expected profit and loss of such a market order is

$$(X_t + \alpha/2 - \eta\alpha) - \left( \frac{1}{1+2\eta}(X_t - 2\eta\alpha) + \frac{2\eta}{1+2\eta}(X_t + \alpha) \right) = \alpha/2 - \eta\alpha.$$

Thus, contrary to the classical efficiency condition of small tick markets which states that the ex post expected cost of a market order should be zero, see for example [43], in the large tick case, it is positive. This means that confronted to this large tick, market takers are ready to lose  $\alpha/2 - \eta\alpha$  in order to obtain liquidity.

As already seen, following Wyart *et al.* [43], the average profit and loss per trade per unit of volume of the market makers is well approximated by  $\alpha/2 - \sigma/\sqrt{M}$ . The gain of the market makers being the loss of the market takers, this leads to

$$\eta\alpha \sim \sigma/\sqrt{M}.$$

Thus, using a theoretical approach inspired by Madhavan *et al.* [31] and Wyart *et al.* [43], we can explain our empirical finding that  $\eta\alpha$  plays the role of spread for large tick assets.

## 4.2 Explanation of microstructure effects

A distinctive feature of high frequency data, particularly of large tick assets, which is the starting point of many econometric models, is the decreasing behavior of the signature plot (2), see among others [3, 4, 6, 20, 38, 45]. If many models aim at reproducing this decreasing shape, there are only few agent based explanations for this phenomenon. Our approach enables us to provide a very simple one.

Recall that the ex post expected cost of a market order is  $\alpha/2 - \eta\alpha$ . This does explain why for large tick assets with average spread close to one tick, the parameter  $\eta$  is systematically smaller than 1/2, which means the signature plot is decreasing. Otherwise we would be in a situation where the cost of market orders is negative and market makers lose money. To avoid that, market makers would naturally increase the spread, which they can always do.

## 5 Changing the tick size

Market designers face the question of choosing a tick value. This is an intricate problem, see [22, 23]. On the one hand, when the tick value is too small, one tick is not really significant, neither for market makers nor for market takers. Therefore, it is very complicated for market makers to choose levels where they should fix their quotes. Furthermore, the order books are very unstable since market participants do not hesitate changing marginally the price of their limit orders in order to gain in priority, which can be very discouraging for market makers. In particular, market participants having only access to a few lines of the order book (typically five), if these lines are not reliable or only provide vanishing liquidity, they may not be able to assess the prices. On the other hand, it is clear that a tick value which is too large prevents the price from moving freely according to the views of market participants whose valuation accuracy for the asset is smaller than one tick.

If the tick value is not satisfying, market platforms have the possibility to change it. Such a modification implies changes on various market quantities (number of trades, spread, liquidity,...). The first issue the platform has to address is those of the desired effects of this change of tick. This is already a difficult question, see Section 5.2. Even in the case where market designers have a clear idea of the situation they want to reach, they still face the problem of the way to reach it. Indeed, it is commonly acknowledged that tick values have to be determined by trial and error and that the success of a change in the tick value can only be assessed ex post, on the basis of the obtained effects. Thus, only few predictive models have been designed in the literature, see for example [21], and the consequences of a change in the tick value have been essentially studied from an empirical point of view, see [1, 5, 7, 10, 11, 12, 18, 19, 28, 37, 42, 44].

## 5.1 The effects of a change in the tick value

We assume we are dealing with a large tick asset. In that case, our approach enables us to forecast *ex ante* the consequences of a change in the tick size on some market quantities, in particular  $\eta$ . To build such predictions, we can rely on market parameters which should be approximately invariant after a change in the tick size, namely the volatility  $\sigma$ , the regression parameter  $p_1$ , possibly the regression parameter  $p_2$ , and the daily traded volume.

However, the daily number of trades  $M$  should not be an invariant quantity. Assume for example that at any time, the average cumulative latent liquidity available up to price  $p$  is of the form  $f(p - \text{midprice})$ , with  $f$  an increasing function, and that each market taker takes a fixed proportion of this liquidity. Then, when the tick size decreases, the available liquidity at the best levels also decreases. The daily traded volume being approximately constant, this implies an increase in the number of transactions  $M$ .

Recall that the market spread is the smallest one achievable so that the market makers profit  $S/2 - \sigma/\sqrt{M}$  is non negative. Therefore, when decreasing the tick size, the market spread remains equal to one tick until the profit and loss of the market makers is equal to zero.

We consider a large tick asset with a starting situation where the market parameters are  $(\alpha_0, \eta_0, M_0)$ . We then change the tick value  $\alpha_0$  to  $\alpha$  and denote by  $(\alpha, \eta, M)$  the new market parameters. We assume we remain in a regime where the spread is approximately equal to one tick, which from the preceding paragraph essentially means that  $M$  is so that  $\alpha/2 - \sigma/\sqrt{M} \geq 0$ . Using the fact that  $\sigma, p_1, p_2$  are invariant together with Equation (4) where the spread is approximated by the tick value we get

$$p_1\eta\alpha\sqrt{M} + p_2\alpha\sqrt{M} = p_1\eta_0\alpha_0\sqrt{M_0} + p_2\alpha_0\sqrt{M_0}.$$

We then consider classical shapes for the function  $f$ . Hence, assuming  $f(x) = c * x$ , leads to

$$\eta = \eta_0 \sqrt{\frac{\alpha_0}{\alpha}} + \frac{p_2}{p_1} \sqrt{\frac{\alpha_0}{\alpha}} - \frac{p_2}{p_1} \quad (5)$$

and assuming  $f(x) = c * \sqrt{x}$  to

$$\eta = \eta_0 \left(\frac{\alpha_0}{\alpha}\right)^{3/4} + \frac{p_2}{p_1} \left(\frac{\alpha_0}{\alpha}\right)^{3/4} - \frac{p_2}{p_1}. \quad (6)$$

The fact that  $p_2$  is constant remaining questionable and its value being small, one can also take  $p_2 = 0$  in (5) and (6).

Therefore, under reasonable assumptions, we are able to forecast the value of  $\eta$  after a change in the tick value. In order to check our relations on real data we use the Bobl contract. The tick value of this asset has been multiplied by two on 2009, June 16. For 12 trading days before 2009, June 16, we give in Figure 5 the estimates of the value of  $\eta$  after the change of the tick value given by Equations (5) and (6).

The results are very satisfying, both assumptions on the latent liquidity leading to good estimates of the future value of  $\eta$ .

## 5.2 Optimal tick size

Defining an optimal tick value is a very complicated issue, see for example [2]. Indeed, different types of market participants can have opposite views on what is a good tick value. For example, market makers might prefer tick values leading to large spread whereas market takers could prefer smaller ticks. Thus, when choosing a tick value, market platforms may favour a given type of participants.

One possibility, of course arguable, is to consider that a spread equal to one tick together

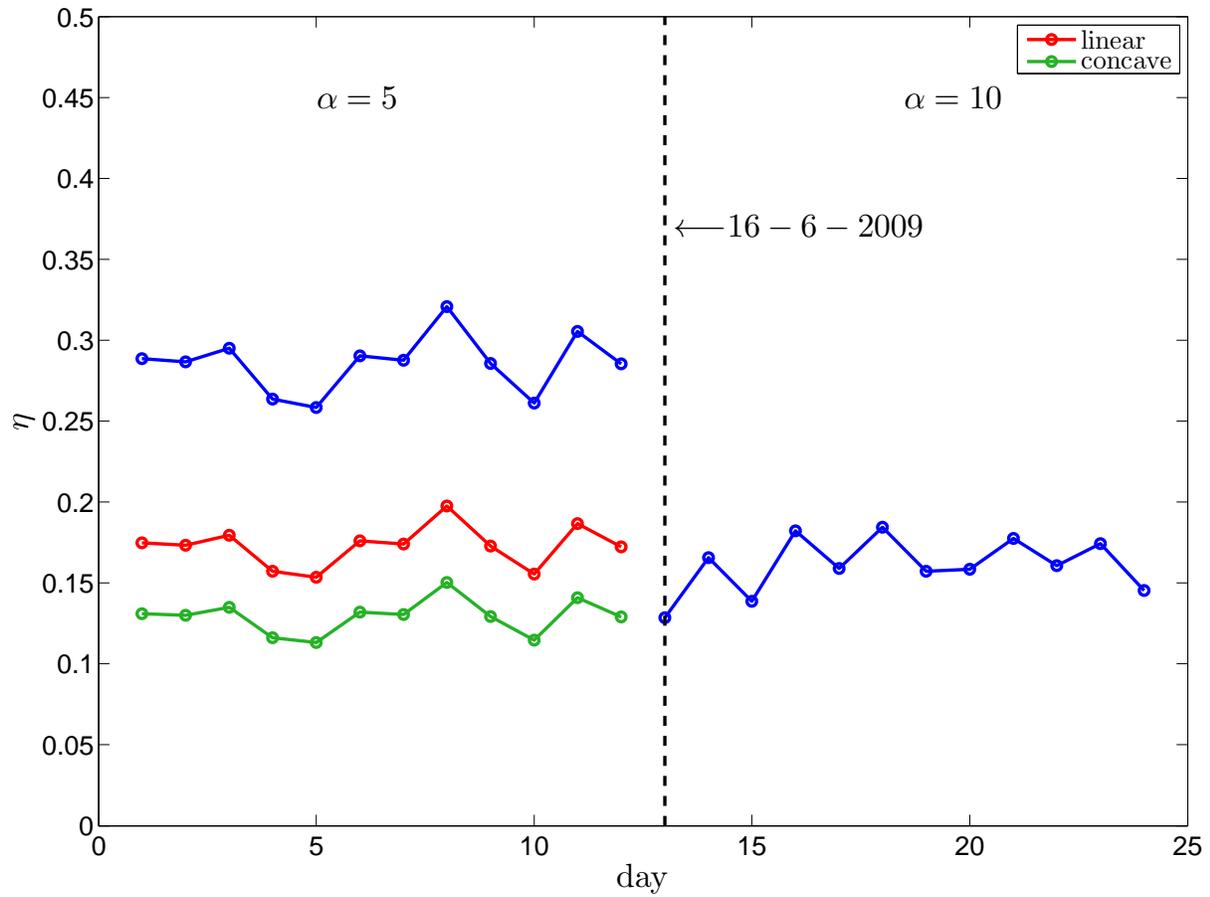


Figure 5: Testing the prediction of  $\eta$  on the Bobl futures. The blue lines show the daily measures of  $\eta$ . The red and green lines are the daily predictions associated to the future tick value.

with  $\eta = 1/2$  is a reasonable situation. Indeed, in that case, market orders and limit orders have both zero cost and the transaction price can be seen as a sampled Brownian motion. In particular, the signature plot is flat, meaning there are no microstructure effects. Therefore  $\eta = 1/2$  might be seen as an *optimal tick size*. Note that starting from a spread equal to one tick, supposing  $\eta$  increases continuously when the tick value decreases, when modifying the tick value, one should be able to reach the state where  $\eta = 1/2$  and the average spread is of order one tick. Indeed, the spread remains equal to one tick as long as  $\alpha/2 - \eta\alpha \geq 0$ . Therefore if  $\alpha^*$  is the largest tick value such that  $\eta = 1/2$ , then for all  $\alpha > \alpha^*$ , market makers make positive profits with a spread of one tick.

Then assuming  $f(x) = c * x$ , from Equation (5),  $\eta = 1/2$  is obtained for

$$\alpha = \alpha_0 \left( \frac{\eta_0 p_1 + p_2}{p_1/2 + p_2} \right)^2 \quad (7)$$

and assuming  $f(x) = c * \sqrt{x}$ , from Equation (6),  $\eta = 1/2$  is obtained for

$$\alpha = \alpha_0 \left( \frac{\eta_0 p_1 + p_2}{p_1/2 + p_2} \right)^{4/3}. \quad (8)$$

Here again, one can also take  $p_2 = 0$  in (7) and (8).

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